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Asymptotic Calculation of a Limit Cycle*

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1. INTRODUCTION

In recent years Prigogine has proposed a model of a biochemical reaction as follows:

$$A \to X$$
$$2X + Y \to 3X$$
$$B + X \to Y + D$$
$$X \to E.$$

The rate equations for this system under various simplifying assumptions give rise to a number of interesting phenomena. (These equations are derived in [7].) When we neglect the effect of diffusion and of temperature, set all rate constants equal to one, and assume that the concentrations of A and B are maintained constant, the equations for the concentrations of X and Y become

$$dX/dt = A - (B + 1) X + X^2 Y \tag{1}$$

$$dY/dt = BX - X^2 Y. \tag{2}$$

This is an autonomous system of ordinary differential equations which has the unique critical point $X = A$, $Y = B/A$. If $B > 1 + A^2$, the critical point is unstable. Under such conditions, phase trajectories starting near the critical point run onto a limit cycle. In [2], Lavenda, Nicolis, and Herschkowitz-Kaufman calculated this limit cycle numerically for the values $B = 77$, $A = 8 \cdot 2$. Since in this case $B \gg 1$, they were able to use phase plane

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arguments to calculate the limit cycle analytically approximately as well. It is a relaxation oscillation, characterized by (nearly) discontinuous jumps. In the present work we calculate a better approximation to the relaxation oscillation using singular perturbation techniques. The calculations are similar to those done for the van der Pol oscillator in Cole's book [1, pp. 38–55]. It will be found that, in contrast to most singular perturbation calculations, the equations to be studied are not derived as the result of a limit process from (1) and (2). (The theory of Pontryagin [6] and Mishchenko [5], which examines discontinuous oscillations in detail using phase plane arguments, does not apply to (1) and (2) without modification.)

2. Transformation to a Second-Order Equation

It is possible to eliminate Y from (1) and (2) by solving (1) for Y in terms of X and dX/dt, and then substituting the values of Y and dY/dt into (2). The result is a second-order equation for X which, however, contains a term $(dX/dt)^2$. This may be removed by the transformation $X = A/u$. (The transformation $X = A/(1 + A\xi)$, employed by Lefever and Nicolis [3], accomplishes the same thing.) The resulting differential equation for u is

$$\frac{d^2u}{dt^2} + \left\{ \frac{A^2}{u^2} + 2u - \nu \right\} \frac{du}{dt} + A^2 \left\{ 1 - \frac{1}{u} \right\} = 0. \tag{3}$$

where $\nu = B + 1 \gg 1$. This equation is clearly equivalent to (1) and (2). We prefer to use (3) since it is of the form

$$u'' + f(u)\, u' + g(u) = 0,$$

and such equations have been intensively studied. The steady state for (3) is $u = 1$. The restoring force $g(u)$ is negative for $u < 1$ and positive for $u > 1$; it tends to A^2 as $u \to +\infty$ and to $-\infty$ as $u \to 0$. The damping $f(u)$ is negative for $u = 1$ but tends to $+\infty$ as $u \to \infty$ and as $u \to 0$. (We always work in the region $u > 0$ since it corresponds to positive concentrations of reactants.) Thus, (3) represents a self-excited oscillation, and the existence of a limit cycle in the unstable case is guaranteed by a minor extension of the Levinson-Smith theorem [4], even when B is not large.

We now proceed to the calculation of the limit cycle in the case $B + 1 = \nu \gg 1$. The perturbation argument is that, in the presence of a small or large parameter, certain terms in the differential equation are negligible in the leading order. In singular perturbations, the negligible terms are different in different regions of the motion. In regions of slow time variation, or outer regions, the acceleration term u'' is negligible. In regions of fast time variation (inner regions or jumps), the restoring force is negligible. Transitional regions occur when representatives from all three terms in (3) are important. We may expect the solution to increase in amplitude when the damping is negative, and decrease when it is positive. Thus, the approximate size of the limit cycle will be determined by the sign changes of the damping. It is seen from (3) that there are two distinct regimes: one where $u \sim \nu$, and one where $u \sim \nu^{-\frac{1}{2}}$. The following regions occur in the description of the motion: the upper outer region, where $u \sim \nu$ and time variations are slow; the upper transition region; the jump down; the lower outer region, where $u \sim \nu^{-\frac{1}{2}}$; the lower transition region; the jump up to the upper outer region. In each region, a separate asymptotic expansion will be necessary. Since the jumps join regions of differing order of magnitude for u, we may expect a certain asymmetry in the equations for the jumps. It will be found that the equations contain the parameter ν explicitly, in contrast to the usual situation for singular perturbations. The balancing of the various terms will require that there be several natural time scales on which the motion progresses.

3. The Upper Outer Solution

The asymptotic expansion in this region will be

$$u \sim \nu u_0 + u_1 + \nu^{-1} u_2 + \cdots .$$

We want to balance the second two terms of (3), so the appropriate time scale is $d/dt \sim \nu^{-2}$. Let $\tau = \nu^{-2} t$. Substitution of the expansion into (3), and equation of coefficients of powers of ν to zero, yields the following hierarchy of equations:

$$(2u_0 - 1)\frac{du_0}{d\tau} + A^2 = 0. \tag{4}$$

$$(2u_0 - 1)\frac{du_1}{d\tau} + 2\frac{du_0}{d\tau} u_1 - \frac{A^2}{u_0} = 0, \tag{5}$$

$$(2u_0 - 1)\frac{du_2}{d\tau} + 2\frac{du_0}{d\tau} u_2 + 2u_1 \frac{du_1}{d\tau} + \frac{A^2 u_1}{u_0^2} = 0. \tag{6}$$

The solution of (4) is found by separation of variables:

$$(2u_0 - 1)\, du_0 = -A^2\, d\tau.$$

Thus,

$$-(u_0 - \tfrac{1}{2})^2 = A^2 \tau + C. \tag{7}$$

Equation (7) represents a family of coaxial parabolas with vertices at $u_0 = \frac{1}{2}$. Since the differential equation is autonomous, we may choose the time origin wherever is convenient. If $C = 0$, then u_0 has its vertical tangent when $\tau = 0$. We choose the upper branch of the parabola since the solution is going to have to jump down to the lower outer region. Thus

$$u_0 = \tfrac{1}{2} + A(-\tau)^{1/2}. \tag{8}$$

Next, (5) may be written, using the chain rule and (4), as

$$\frac{du_1}{du_0} + \frac{2}{2u_0 - 1} u_1 = -\frac{1}{u_0};$$

that is,

$$\frac{1}{u_0 - \frac{1}{2}} \frac{d}{du_0}\left[\left(u_0 - \frac{1}{2}\right) u_1\right] = -\frac{1}{u_0}.$$

Therefore,

$$u_1 = \frac{1}{u_0 - \frac{1}{2}} \left(-u_0 + \frac{1}{2} \log u_0 + A_1\right),$$

where A_1 is a constant of integration, to be determined from matching. Using (8) we obtain

$$u_1 = \frac{1}{A(-\tau)^{1/2}} \left\{A_1 - \frac{1}{2}(1 + \log 2) - A(-\tau)^{1/2} + \frac{1}{2}\log(1 + 2A(-\tau)^{1/2})\right\}. \tag{9}$$

As $\tau \to 0^-$, u_1 has the expansion

$$u_1 \sim \frac{1}{A(-\tau)^{1/2}} \left\{A_1 - \frac{1}{2}(1 + \log 2)\right\} - A(-\tau)^{1/2} - \frac{8}{3}\tau + O((-\tau)^{3/2}). \tag{10}$$

4. The Lower Outer Solution

In this region the asymptotic expansion will be $u \sim \nu^{-\frac{1}{2}} w_0 + \nu^{-1} w_1 + \cdots$, and in order to balance the second two terms of (3) we let $d/dt \sim 1$. Then the hierarchy of equations is

$$\left(\frac{A^2}{w_0^2} - 1\right) \frac{dw_0}{dt} - \frac{A^2}{w_0} = 0, \tag{11}$$

$$\left(\frac{A^2}{w_0^2} - 1\right) \frac{dw_1}{dt} + w_1 \left(\frac{A^2}{w_0^2} - \frac{2A^2}{w_0^3} \frac{dw_0}{dt}\right) + A^2 = 0. \tag{12}$$

The solution to (1) is

$$A^2 \log w_0 - \tfrac{1}{2} w_0^2 = A^2 t + \mathscr{C}.$$

Since the solution is going to have to jump up to the upper outer region, we choose the lower branch of this function; it has a vertical tangent at $w_0 = A$. Call the time at which the tangent is vertical t_j ($= t_j(\nu)$). Thus

$$A^2 \log w_0 - A^2 \log A - \tfrac{1}{2}w_0^2 + \tfrac{1}{2}A^2 = A^2(t - t_j).$$

Let $\hat{t} = t - t_j$; then the lower outer limit will have $\hat{t}$ fixed. Thus

$$A^2 \log w_0 - A^2 \log A - \tfrac{1}{2}w_0^2 + \tfrac{1}{2}A^2 = A^2\hat{t}. \tag{13}$$

Next, (12) may be written

$$\frac{dw_1}{dw_0} + \left(\frac{1}{w_0} - \frac{2A^2}{w_0(A^2 - w_0^2)}\right) w_0 - 1 = 0,$$

or

$$\frac{d}{dw_0}\left(\frac{A^2 - w_0^2}{w_0} w_1\right) = \frac{w_0^2 - A^2}{w_0}.$$

Therefore,

$$w_1 = \frac{1}{A^2 - w_0^2}\left\{\frac{1}{2} w_0^3 - A^2 w_0 \log w_0 + \mathscr{D}_1 w_0\right\}, \tag{14}$$

where $\mathscr{D}_1$ is a constant of integration.

We shall be interested in the expansions of w_0 and w_1 as $\hat{t} \to -t_j$ and as $\hat{t} \to 0^-$. We postpone consideration of the first limit for a while, but can write immediately that as $\hat{t} \to 0^-$,

$$w_0 \sim A(1 - (-\hat{t})^{1/2} - \frac{1}{6}\hat{t} + \cdots) \tag{15}$$

and

$$w_1 \sim \frac{\tfrac{1}{2}A^2 - A^2 \log A + \mathscr{D}_1}{2A}\frac{1}{\hat{t}^{1/2}} + \frac{1}{12}A - \frac{1}{6}A \log A + \frac{1}{6}\frac{\mathscr{D}_1}{A} + O(-\hat{t})^{1/2}. \tag{16}$$

5. The Inner Solutions

The appropriate time scale to balance the first two terms of (3) is $d/dt \sim \nu$. In the original variables, the leading equation for the jumps is thus (in both cases)

$$\frac{d^2u}{dt^2} + \left\{\frac{A^2}{u^2} + 2u - \nu\right\}\frac{du}{dt} = 0, \tag{17}$$

or

$$\frac{du}{dt} + u^2 - \nu u - \frac{A^2}{u} = C. \tag{18}$$

Figure 1 shows the phase trajectories of Eq. (17). The line $u' = 0$ is singular. It is seen that there are two distinguished trajectories QR and ST which correspond to a jump up between two finite values of u, and to a jump down. Other trajectories start at $u = 0$, $u' = \infty$ or $u = \infty$, $u' = -\infty$ and are not of interest. The points Q and S are double zeros of the cubic $u^2 - \nu u - A^2/u = C$, while R and T are simple zeros. For the jump down it will be necessary that S match with the jump-off point $u = \frac{1}{2}\nu$ of the upper outer solution, at least to $o(1)$. This can be done by proper choice of C. For the jump up, Q must match with the jump-off point $u = A\nu^{-\frac{1}{2}}$. This can be done by another choice of C.

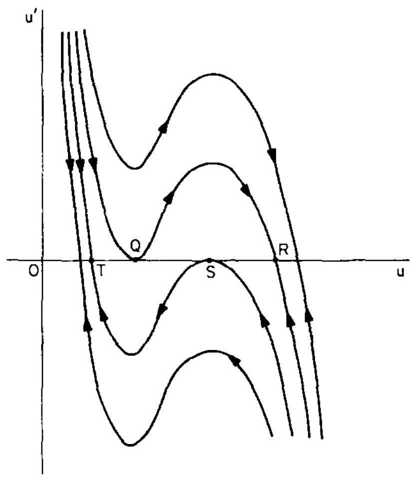

FIG. 1. Phase trajectories of Eq. (17).

An important point is that in the jump down, when (at least at first) $u \sim \nu$, it is still necessary to include the term A^2/u^2 in Eq. (17). If it is omitted, the phase trajectories are no longer cubics but parabolas, and these have completely the wrong behavior for jumps. In the jump up, the term $2u$ must be included for the same reason. Although these terms are quantitatively negligible at the beginning of the jumps, they are important at the end. Thus, it will be necessary to treat them as being of honorary order ν. The asymmetry introduced is not found in the similar equations for the van der Pol oscillator. In this case, the inclusion of the extra terms implies that the inner equations are not obtained as the result of a limit process, and the equations in the hierarchy all contain ν explicitly.

For the jump down we try $u \sim \nu g_0(\nu, \bar{t}) + \beta_1(\nu)\, g_1(\nu, \bar{t}) + \beta_2(\nu)\, g_2(\nu, \bar{t}) + \cdots$, where $\bar{t} = \nu(t - \delta(\nu))$. There is a possible time shift, because only $d\bar{t}/dt$ is determined from balancing considerations. The coefficients $\beta_1(\nu)$, $\beta_2(\nu)$,..., and $\delta(\nu)$ are to be found from matching. Since we treat A^2/u^2 as being of honorary order ν, the hierarchy of equations obtained from (3) is

$$\frac{d^2g_0}{d\bar{t}^2} + \left[\frac{A^2}{\nu^3 g_0^2} + 2g_0 - 1\right]\frac{dg_0}{d\bar{t}} = 0, \qquad (19)$$

$$\frac{d^2g_1}{d\bar{t}^2} + \left[\frac{A^2}{\nu^3 g_0^2} + 2g_0 - 1\right]\frac{dg_1}{d\bar{t}} + \left[2 - \frac{2A^2}{\nu^3 g_0^3}\right]\frac{dg_0}{d\bar{t}}\, g_1 = 0. \qquad (20)$$

Integrating (19) once, we have

$$\frac{dg_0}{d\bar{t}} + \left[-\frac{A^2}{\nu^2 g_0} + g_0^2 - g_0\right] = C_0\,.$$

As $\bar{t} \to -\infty$, we want g_0 to tend $\alpha = \frac{1}{2} + o(1)$ and $dg_0/d\bar{t}$ to tend to zero; α must be a double root of $-(A^2/(\nu^3 g_0)) + g_0^2 - g_0 = C_0$. Call the other root β; then (21) becomes

$$\frac{dg_0}{d\bar{t}} = -\frac{(g_0 - \alpha)^2\,(g_0 - \beta)}{g_0} = (\alpha - g_0)^2\left(\frac{\beta}{g_0} - 1\right). \qquad (22)$$

The condition that α be a double root requires that

$$2\alpha^3 - \alpha^2 + (A^2/\nu^3) = 0.$$

Three iterations using Newton's method on this equation yield

$$\alpha = \frac{1}{2}\left(1 - \frac{4A^2}{\nu^3} + \frac{32A^4}{\nu^6} + O(\nu^{-9})\right) \qquad (23)$$

and

$$\beta = 1 - 2\alpha = \frac{4A^2}{\nu^3}\left(1 - \frac{8A^2}{\nu^3} + O(\nu^{-6})\right). \qquad (24)$$

From Eq. (22) we see that the decay of g_0 to α as $\bar{t} \to -\infty$ is only algebraic, whereas the decay to β as $\bar{t} \to +\infty$ is exponential. As $\bar{t} \to -\infty$, g_0 has the expansion

$$g_0 = \alpha + \frac{A_1}{\bar{t}} + A_{1,2}\frac{\log(-\bar{t})}{\bar{t}^2} + \frac{A_2}{\bar{t}^2} + A_{2,3}\frac{\log^2(-\bar{t})}{\bar{t}^3} + \cdots, \qquad (25)$$

where

$$A_1 = \left(1 - \frac{\beta}{\alpha}\right)^{-1} = 1 + \frac{8A^2}{\nu^3} + \cdots,$$

$$A_{1,2} = -\frac{\beta}{\alpha^2}\left(1 - \frac{\beta}{\alpha}\right)^{-3} = -\frac{16A^2}{\nu^3}\left(1 - \frac{24A^2}{\nu^3} + \cdots\right),$$

$$A_2 = \tfrac{1}{2}A_{1,2}\,.$$

As $\bar{t} \to +\infty$, g_0 has the expansion

$$\begin{aligned} g_0 &= \beta + O\left(\exp\left(-\frac{(\beta-\alpha)^2}{\beta}\bar{t}\right)\right) \\ &= \frac{4A^2}{\nu^3}\left(1 - \frac{8A^2}{\nu^3} + \cdots\right) + \text{transcendentally small terms (TST).} \end{aligned} \tag{26}$$

Next, (20) implies

$$\frac{d}{d\bar{t}}\left\{\frac{dg_1}{d\bar{t}} + g_1\left(\frac{A^2}{\nu^3 g_0{}^2} + 2g_0 - 1\right)\right\} = 0.$$

Certainly $g_1 = h_1(dg_1/d\bar{t})$, where h_1 is a constant of integration, is a solution. The general solution will be the sum of this and the particular integral of

$$\frac{dg_1}{d\bar{t}} + \left(\frac{A^2}{\nu^3 g_0{}^2} + 2g_0 - 1\right) = k_1\,. \tag{27}$$

We can find the leading order contribution to g_1 as $\bar{t} \to \pm\infty$. As $\bar{t} \to -\infty$, Eq. (27) is asymptotic to

$$\frac{dg_1}{d\bar{t}} + \left\{\frac{1}{\bar{t}}(2 + \cdots) + \frac{\log(-\bar{t})}{\bar{t}^2}\left(-32\frac{A^2}{\nu^3} + \cdots\right) + \frac{1}{\bar{t}^2}\left(\frac{32A^2}{\nu^3} + \cdots\right)\right\} g_1$$
$$= k_1\,.$$

Thus, the particular integral is asymptotic to

$$g_{1p} = k_1\left\{\left(\frac{1}{3} + O(\nu^{-6})\right)\bar{t} + \left(\frac{16}{3}\frac{A^2}{\nu^3} + \cdots\right)\log(-\bar{t}) + \frac{8A^2}{\nu^3} + \cdots + O\left(\frac{\log(-\bar{t})}{\bar{t}}\right)\right\},$$

and hence, as $\bar{t} \to -\infty$,

$$g_1 \sim \tfrac{1}{3}k_1\bar{t} + \cdots.$$

The homogeneous solution does not enter until later in the expansion. As yet the constant k_1 is not known. As $\bar{t} \to +\infty$, (27) is asymptotic to

$$\frac{dg_1}{d\bar{t}} + \left(\frac{A^2}{\nu^3\beta^2} + 2\beta - 1 + \text{TST}\right) g_1 = k_1\,.$$

Thus, as $\bar{t} \to +\infty$,

$$g_1 = \frac{16A^2}{\nu^3}k_1 + \text{TST}.$$

The expansion for the jump up will be $u \sim \nu^{-\frac{1}{2}} z_0 + \mu_1(\nu) z_1 + \cdots$, with the time $\tilde{t} = \nu(\hat{t} - \chi(\nu))$. As mentioned above, it will be necessary to treat the term $2u$ as being of order ν. The hierarchy of equations that results is

$$\frac{d^2 z_0}{d\tilde{t}^2} + \left[\frac{A^2}{z_0^{\,2}} + \frac{2z_0}{\nu^{3/2}} - 1\right] \frac{dz_0}{d\tilde{t}} = 0, \tag{28}$$

$$\frac{d^2 z_1}{d\tilde{t}^2} + \left[\frac{A^2}{z_0^{\,2}} + \frac{2z_0}{\nu^{3/2}} - 1\right] \frac{dz_1}{d\tilde{t}} - \frac{dz_0}{d\tilde{t}} \left[\frac{2}{\nu^{3/2}} - \frac{2A^2}{z_0^{\,3}}\right] z_1 = 0. \tag{29}$$

One integration of (28) yields

$$\frac{dz_0}{d\tilde{t}} + \left[-\frac{A^2}{z_0} + \frac{z_0^{\,2}}{\nu^{3/2}} - z_0\right] = \mathscr{C}. \tag{30}$$

As $\tilde{t} \to -\infty$ we want $z_0 = A + o(1)$ and $dz_0/d\tilde{t} \to 0$; the value ϕ that z_0 attains must be a double root of

$$-\frac{A^2}{z_0} + \frac{z_0^{\,2}}{\nu^{3/2}} - z_0 = \mathscr{C}.$$

If the other root is called ψ, then (30) becomes

$$\frac{dz_0}{d\tilde{t}} = \frac{1}{\nu^{3/2}} \frac{(z_0 - \phi)^2 (\psi - z_0)}{z_0}, \tag{31}$$

and ϕ must satisfy the equation, $2\phi^3 - \nu^{3/2}\phi^2 + \nu^{3/2}A^2 = 0$. Three iterations using Newton's method yield

$$\phi = A\left(1 + \frac{A^2}{\nu^{3/2}} + \frac{5}{2}\frac{A^2}{\nu^2} + O(\nu^{-9/2})\right),$$

$$\psi = \nu^{3/2} - 2\phi = \nu^{3/2}\left(1 - \frac{2A}{\nu^{3/2}} - \frac{2A^2}{\nu^3} - \frac{5A^3}{\nu^{9/2}} + O(\nu^{-6})\right).$$

From (31) it is found that as $\tilde{t} \to -\infty$,

$$z_0 = \phi + \frac{\mathscr{g}_1}{\tilde{t}} + \mathscr{g}_{1,2} \frac{\log(-\tilde{t})}{\tilde{t}^2} + \frac{\mathscr{g}_2}{\tilde{t}^2} + \cdots, \tag{32}$$

where

$$\mathscr{g}_1 = -\nu^{3/2}\left(\frac{\psi}{\phi} - 1\right)^{-1} = -A\left(1 + \frac{4A}{\nu^{3/2}} + \cdots\right),$$

$$\mathscr{g}_{1,2} = \nu^3 \frac{\psi}{\phi^2}\left(\frac{\psi}{\phi} - 1\right)^{-3} = A\left(1 + \frac{8A}{\nu^{3/2}} + \cdots\right),$$

$$\mathscr{g}_2 = \tfrac{1}{2}\mathscr{g}_{1,2}\,.$$

As $\tilde{t} \to \infty$.

$$z_0 = \psi + \mathrm{TST} = \nu^{3/2}\left(1 - \frac{2A}{\nu^{3/2}} - \frac{2A^2}{\nu^3} + \cdots\right) + \mathrm{TST}. \tag{33}$$

Next, (29) implies

$$\frac{dz_1}{d\tilde{t}} + \left(\frac{A^2}{{z_0}^2} + \frac{2z_0}{\nu^{3/2}} - 1\right) z_1 = \mathscr{B}_1, \tag{34}$$

where $\mathscr{B}_1$ is a constant of integration. As $\tilde{t} \to -\infty$, we find by the same method as for the jump down,

$$z_1 = \mathscr{B}_1 \left[\frac{1}{3}(1 + O(\nu^{-3}))\,\tilde{t} + \frac{2}{3}\left(1 + \frac{4A}{\nu^{3/2}} + \cdots\right)\log(-\tilde{t}) - \frac{1}{2}\left(1 + \frac{4A}{\nu^{3/2}}\right)\right] + O\left(\frac{\log(-\tilde{t})}{\tilde{t}^2}\right). \tag{35}$$

As $\tilde{t} \to +\infty$,

$$\begin{aligned} z_1 &= \frac{\mathscr{B}_1}{(A^2/\psi^2) + (2\psi/\nu^{3/2}) - 1} + \mathrm{TST} \\ &= \mathscr{B}_1\left(1 + \frac{4A}{\nu^{3/2}} + \frac{19A^2}{\nu^3} + \cdots\right) + \mathrm{TST}. \end{aligned} \tag{36}$$

6. Leading Order Calculation of the Period

Knowledge of the ends of the jumps enables us to make a first approximation to the period at this stage. The jumps themselves are treated as being instantaneous, and the period is the sum of the time taken for the upper outer solution to get from $u = \nu^{-\frac{1}{2}}\psi$ to $u = \frac{1}{2}\nu$ and the time taken for the lower outer solution to get from $u = \nu\beta$ to $u = \nu^{-\frac{1}{2}}A$. Substitution of these values into (8) and (13) yields

$$T = \frac{\nu^2}{4A^2}\left(1 - \frac{8A}{\nu^{3/2}} + \cdots\right) + \frac{3}{2}\log\nu - \log A - 2\log 2 + \cdots.$$

This formula agrees to leading order with the one calculated by phase plane methods by Lavenda, Nicolis, and Herschkowitz–Kaufman [2]. The form of the solution $u(t)$ is shown in Fig. 2, and the concentration $X = A/u$ is shown in Fig. 3.

Although the inner and outer solutions match to leading order, the algebraic decay of the inner solution as $\tilde{t} \to -\infty$ and as $\tilde{t} \to -\infty$ means that higher order matching cannot be carried out. A more detailed computation of the solution, and thus of the period, requires the introduction of transition expansions. The upper transition solution must match the upper outer solution as $\tau \to 0^-$ and the jump down as $\bar{t} \to -\infty$. The lower transition solution must match the lower outer solution as $\hat{t} \to 0^-$ and the jump up as $\tilde{t} \to -\infty$. (Since the decay of the inner solutions is exponential as $\tilde{t} \to +\infty$ and $\bar{t} \to +\infty$, there is no trouble in matching them to the outer solutions there.) Each transition solution must contain representatives from all three terms in Eq. (3).

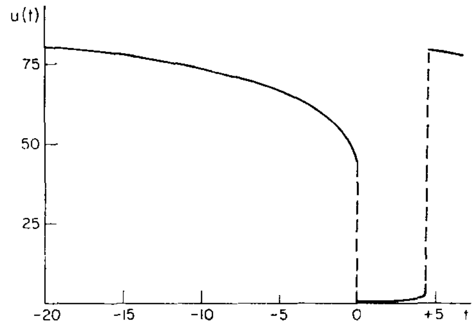

FIG. 2. One oscillation of $u(t)$.

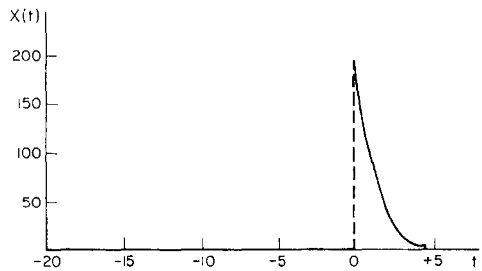

FIG. 3. One oscillation of $X = A/u$. The concentration of X is nearly zero throughout most of the oscillation.

7. The Upper Transition Solution

The necessary time scaling in this region is $d/dt \sim 1$, say $t^* = t - \rho(\nu)$, and the expansion is of the form $u \sim \frac{1}{2}\nu + f_1 + \nu^{-1} f_2 + \cdots$. The hierarchy of equations is

$$\frac{d^2 f_1}{dt^{*2}} + 2 f_1 \frac{df_1}{dt^*} + A^2 = 0, \tag{37}$$

$$\frac{d^2 f_2}{dt^{*2}} + 2 f_1 \frac{df_2}{dt^*} + 2 \frac{df_1}{dt^*} f_2 - 2A^2 = 0. \tag{38}$$

One integration of (37) yields

$$\frac{df_1}{dt^*} + f_1^2 + A^2 t^* = 0, \tag{39}$$

where the constant of integration has been absorbed into $\rho(\nu)$. If we let $f_1 = V'/V = d/dt^* \log V$, then (39) becomes

$$\frac{d^2 V}{dt^{*2}} + A^2 t^* V = 0. \tag{40}$$

Substituting $z = \theta t^*$, where $\theta^2 = A^2/\theta$, we find that (40) becomes

$$\frac{d^2 V}{dz^2} + zV = 0,$$

which is Airy's equation for negative argument and has the solution

$$V = M(-z)^{1/2} K_{1/3}(\tfrac{2}{3}(-z)^{3/2}) + N(-z)^{1/2} I_{1/3}(\tfrac{2}{3}(-z)^{3/2}).$$

$I_{\frac{1}{3}}$ and $K_{\frac{1}{3}}$ are modified Bessel functions. As $t^* \to -\infty$ (i.e., $z \to -\infty$) they have the expansions

$$K_{1/3}(\tfrac{2}{3}(-z)^{3/2}) = \tfrac{1}{2}(3\pi)^{1/2}(-z)^{-3/4} \exp[-\tfrac{2}{3}(-z)^{3/2}]\{1 - \tfrac{5}{48}(-z)^{-3/2} + \cdots\},$$

$$I_{1/3}(\tfrac{2}{3}(-z)^{3/2}) = \tfrac{1}{2}(3/\pi)^{1/2}(-z)^{-3/4} \exp[\tfrac{2}{3}(-z)^{3/2}]\{1 + \tfrac{5}{48}(-z)^{3/2} + \cdots\}. \tag{41}$$

In order to match to the upper outer solution, we need to choose the solution of (39) that behaves as $+(-t^*)^{\frac{1}{2}}$ as $\tau \to 0^-$. This can happen only if $N = 0$. This choice uses up one of the constants of integration; the other is in $\rho(\nu)$. Since $f_1 = V'/V$, we can choose $M = 1$ and obtain

$$\begin{aligned} V(z) &= (-z)^{1/2} K_{1/3}(\tfrac{2}{3}(-z)^{3/2}) && \text{for } z < 0 \\ &= (\pi/3^{1/2})\, z^{(1/2)}\{J_{1/3}(\tfrac{2}{3}z^{3/2}) + J_{-1/3}(\tfrac{2}{3}z^{3/2})\} && \text{for } z > 0. \end{aligned} \tag{42}$$

As $t^* \to -\infty$, f_1 therefore has the expansion

$$f_1 = A(-t^*)^{1/2} - \frac{1}{4t^*} - \frac{5}{32A}(-t^*)^{-5/2} + O((-t^*)^{-7/2}), \tag{43}$$

found from (41) and (42). To match to the jump down we take the limit $z \to \omega_0$, where ω_0 is the first zero of the Airy function. It is a simple zero with the approximate value $\omega_0 = 2.338$. As z tends to ω_0,

$$V(z) \propto -(z - \omega_0) + \tfrac{1}{6}\omega_0(z - \omega_0)^3 + O((z - \omega_0)^5). \tag{44}$$

As z tends to ω_0, $t^* = A^{-\frac{2}{3}}z$ tends to $\theta_0 = A^{-\frac{2}{3}}\omega_0$. So as $t^* \to \theta_0$,

$$f_1 = \frac{1}{t^* - \theta_0} - \tfrac{1}{3}A^{4/3}\omega_0(t^* - \theta_0) + O((t^* - \theta_0)^3). \tag{45}$$

Next, (38) becomes upon integration

$$\frac{df_2}{dt^*} + 2f_1 f_2 = 2A^2 t^* + C_2;$$

that is,

$$V^2 f_2 = \int_{-\infty}^{t^*} 2A^2\lambda V^2(\lambda)\, d\lambda + \int_{-\infty}^{t^*} C_2 V^2(\lambda)\, d\lambda + D_2 .$$

To prevent exponential growth of f_2 as $t^* \to -\infty$, we must choose $D_2 = 0$. Thus

$$f_2(t^*) = \frac{C_2}{V^2} \int_{-\infty}^{t^*} V^2(\lambda)\, d\lambda + \frac{2A^2}{V^2} \int_{-\infty}^{t^*} \lambda V^2(\lambda)\, d\lambda. \tag{46}$$

Using (41) and integrating by parts, we find that as $t^* \to -\infty$,

$$f_2 = -A(-t^*)^{1/2} + \frac{C_2}{2A} \left\{ \frac{1}{(-t^*)^{1/2}} + \frac{11}{12A} t^{*-2} \right\} + O((-t^*)^{-5/2}). \tag{47}$$

Using (45) we find that as $t^* \to \theta_0$,

$$\begin{aligned} f_2 = \frac{1}{(t^* - \theta_0)^2} A^{-4/3} &\left\{ C_2 \int_{-\infty}^{\theta_0} V^2(\lambda)\, d\lambda + 2A^2 \int_{-\infty}^{\theta_0} \lambda V^2(\lambda)\, d\lambda \right\} \\ &\times \{1 + \tfrac{1}{3}A^{4/3}\omega_0(t^* - \theta_0)^2 + \cdots\}. \end{aligned} \tag{48}$$

8. The Lower Transition Solution

In this region it is necessary to use the expansion

$$u \sim \nu^{-1/2}A + \nu^{-5/6}y_1 + \nu^{-7/6}y_2 + \cdots$$

and to take $d/dt \sim \nu^{2/3}$, so that $t^+ = \nu^{2/3}(\hat{t} - \lambda(\nu))$. The hierarchy of equations is

$$\frac{d^2y_1}{dt^{+2}} - \frac{2}{A} y_1 \frac{dy_1}{dt^+} - A = 0, \tag{49}$$

$$\frac{d^2y_2}{dt^{+2}} + \left[-\frac{2}{A} y_2 + \frac{3y_1^2}{A^2} \right] \frac{dy_1}{dt^+} - \frac{2}{A} y_1 \frac{dy_2}{dt^+} - y_1 = 0. \tag{50}$$

One integration of (49) yields

$$\frac{dy_1}{dt^+} - \frac{1}{A} y_1^2 - At^+ = 0, \tag{51}$$

where once again the constant of integration has been absorbed into $\lambda(\nu)$. If we let $y_1 = -AV'/V$, then (51) becomes

$$\frac{d^2 V}{dt^{+2}} + t^+ V = 0.$$

Thus

$$V(t^+) = M(-t^+)^{1/2} K_{1/3}(\tfrac{2}{3}(-t^+)^{3/2}) + N(-t^+)^{1/2} I_{1/3}(\tfrac{2}{3}(-t^+)^{3/2}).$$

We want the solution of (51) that behaves as $-(-\hat{t})^{1/2}$ as $\hat{t} \to 0^-$. This forces $N = 0$, and as before we may also choose $M = 1$. Thus

$$\begin{aligned} V(t^+) &= (-t^+)^{1/2} K_{1/3}(\tfrac{2}{3}(-t^+)^{3/2}) && \text{if} \quad t^+ < 0 \\ &= (\pi/3^{1/2})\,(t^+)^{1/2}\,\{J_{1/3}(\tfrac{2}{3}t^{+3/2}) + J_{-1/3}(\tfrac{2}{3}t^{+3/2})\} && \text{if} \quad t^+ > 0. \end{aligned}$$

As $t^+ \to \infty$, y_1 has the expansion

$$y_1 = -A(-t^+)^{1/2} + \frac{A}{4t^+} + \frac{5A}{32}(-t^+)^{-5/2} + O(1 - t^+)^{-7/2}), \tag{52}$$

and as $t^+ \to \omega_0$, y_1 has the expansion

$$y_1 = -A\left[\frac{1}{t^+ - \omega_0} - \frac{1}{3}\omega_0(t^+ - \omega_0) + O((t^+ - \omega_0)^2)\right]. \tag{53}$$

Next, one integration of (50) yields

$$\frac{dy_2}{dt^+} + 2\left(\frac{d}{dt^+}\log V\right) y_2 + \frac{y_1^3}{A^2} = -A\log V + \mathscr{E}_2,$$

or

$$\frac{1}{V^2}\frac{d}{dt^+}(V^2 y_2) - \frac{AV'^3}{V^3} = -A\log V + \mathscr{E}_2. \tag{54}$$

A further integration together with the use of (51) and integration by parts gives

$$y_2 = \frac{\mathscr{F}_2}{V^2} + \frac{dy_1}{dt^+}\left(\frac{\mathscr{E}_2}{A} + \log V\right) - \frac{2A}{V^2}\int_{-\infty}^{t^+} V^2 \log V\, d\lambda + \frac{1}{2}\frac{y_1^2}{A}. \tag{55}$$

To prevent exponential growth of y_2 as $t^+ \to -\infty$, it is necessary to choose $\mathscr{F}_2 = 0$. As $t^+ \to -\infty$, y_2 then has the expansion

$$\begin{aligned} y_2 = A \Big\{ \frac{1}{6} t^+ + \frac{1}{8} \frac{\log(-t^+)}{(-t^+)^{1/2}} \\ + \left[\frac{1}{2} \frac{\mathscr{E}_2}{A} - \frac{1}{2} \log \frac{(3\pi)^{1/2}}{2} + \frac{4}{9} \right] \frac{1}{(-t^+)^{1/2}} + \cdots \Big\}, \end{aligned} \tag{56}$$

and as $t^+ \to \omega_0$, y_2 has the expansion

$$\begin{aligned} y_2 = A \frac{\log(\omega_0 - t^+)}{(t^+ - \omega_0)^2} \\ + \frac{1}{(t^+ - \omega_0)^2} \Big\{ \mathscr{E}_2 - 2A \int_{-\infty}^{\omega_0} V^2 \log V \, d\lambda + \frac{1}{2} A + \cdots \Big\}. \end{aligned} \tag{57}$$

9. Matching of the Various Expansions

Since the expressions for the functions to be matched and their expansions in the various time limits are complicated, we find it safer to go explicitly through the intermediate limit procedure in every case.

To match the upper outer and transition solutions, we consider the limit $\tau \to 0^-$, $t^* \to -\infty$. We let $t_\eta = (t - \rho(\nu))/\eta(\nu) < 0$ be fixed, where $1 \ll \eta \ll \nu^2$ as $\nu \to \infty$. Thus, $t^* = \eta t_\eta$, $\tau = \nu^{-2}(\eta t_\eta + \rho)$. It may be assumed that $\rho \ll \eta$. From (8) and (10), the expansion of the upper outer solution in the intermediate variables is

$$\begin{aligned} u &= \frac{\nu}{2} + A(-\eta t_\eta - \rho)^{1/2} + \frac{\nu}{A(-\eta t_\eta - \rho)^{1/2}} \Big\{ A_1 - \frac{1}{2}(1 + \log 2) \Big\} \\ &\quad - \frac{A}{\nu}(-\eta t_\eta - \rho)^{1/2} + \cdots \\ &= \frac{\nu}{2} + A(-\eta t_\eta)^{1/2} - \frac{1}{2} \frac{A\rho}{(-\eta t_\eta)^{1/2}} + \nu \frac{A_1 - \frac{1}{2}(1 + \log 2)}{A} \frac{1}{(-\eta t_\eta)^{1/2}} \\ &\quad - \frac{A}{\nu}(-\eta t_\eta)^{1/2} + \cdots . \end{aligned} \tag{58}$$

From (43) and (47), the expansion of the upper transition solution in the intermediate variables is

$$u = \frac{\nu}{2} + A(-\eta t_\eta)^{1/2} - \frac{A}{\nu}(-\eta t_\eta)^{1/2} + \cdots . \tag{59}$$

Both ρ and A_1 should be determined from a comparison of (58) and (59). Thus, we must choose $\rho = 0$ and $A_1 = \frac{1}{2}(1 + \log 2)$. The terms omitted vanish faster than those matched.

To match the upper transition solution and the jump down, we take the limit $t^* \to \theta_0$, $\bar{t} \to -\infty$. We let $\nu^{-1} \ll \eta \ll 1$ and $t_\eta < 0$ be fixed. Then

$$\bar{t} = \nu(t - \delta(\nu)) = \nu(t^* - \delta(\nu)) = \nu\eta t_\eta .$$

Thus, $t^* = \eta t_\eta + \delta$ and $t^* - \theta_0 = \eta t_\eta + \sigma(\nu)$, where $\sigma(\nu) = \delta(\nu) - \theta_0$. We may assume $\sigma \gg \eta$. In the intermediate variables the upper transition solution is

$$\begin{aligned} u &= \frac{\nu}{2} + \frac{1}{\eta t_\eta + \sigma} - \frac{1}{3} A^{4/3}\omega_0(\eta t_\eta + \sigma) + \cdots \\ &\quad + \frac{1}{\nu(\eta t_\eta + \sigma)^2} A^{-4/3} \left\{ C_2 \int_{-\infty}^{\theta_0} V^2\, d\lambda + 2A^2 \int_{-\infty}^{\theta_0} \lambda V^2\, d\lambda \right\} \\ &= \frac{\nu}{2} + \frac{1}{\eta t_\eta} - \frac{\sigma}{(\eta t_\eta)^2} + \cdots - \frac{1}{3} A^{4/3}\omega_0 \eta t_\eta + \cdots \\ &\quad + \frac{1}{\eta(\eta t_\eta)^2} A^{-4/3} \left\{ C_2 \int_{-\infty}^{\theta_0} V^2\, d\lambda + 2A^2 \int_{-\infty}^{\theta_0} \lambda V^2\, d\lambda \right\}, \end{aligned} \tag{60}$$

and the inner solution is

$$\begin{aligned} u &= \frac{\nu}{2}\left(1 - \frac{4A^2}{\nu^3} + \cdots\right) + \left(1 + \frac{8A^2}{\nu^3} + \cdots\right)\frac{1}{\eta t_\eta} + \cdots \\ &\quad + \beta_1(\nu)\, \tfrac{1}{3} k_1 \nu\eta t_\eta + \cdots . \end{aligned} \tag{61}$$

Comparison of (60) and (61) shows that we must take

$$\beta_1 = \nu^{-1}, \qquad k_1 = -A^{4/3}\omega_0 , \qquad \sigma = 0,$$

and

$$C_2 = -2A^2 \int_{-\infty}^{\theta_0} \lambda V^2\, d\lambda \Big/ \int_{-\infty}^{\theta_0} V^2\, d\lambda.$$

Thus, the inner expansion is $u \sim \nu g_0 + \nu^{-1} g_1 + \cdots$, and as $\bar{t} \to +\infty$,

$$u = \frac{4A^2}{\nu^3}\left(1 - \frac{8A^2}{\nu^3} + \cdots\right) - \frac{16A^2}{\nu^3} A^{4/3}\omega_0 + \text{TST}. \tag{62}$$

The time shift $\delta(\nu)$ is equal to $\theta_0 = A^{-2/3}\omega_0$.

The limit to be taken in matching the lower outer and transition solutions is $\hat{t} \to 0^-$, $t^+ \to -\infty$. We introduce $\eta(\nu)$ where $\nu^{-2/3} \ll \eta \ll 1$, let $t_\eta < 0$ be fixed, and put $t^+ = \nu^{2/3}(\hat{t} - \lambda(\nu)) = \nu^{2/3}\eta t_n$. Then $\hat{t} = \eta t_n + \lambda$. In the intermediate variables the outer expansion is

$$\begin{aligned} u &= \frac{A}{\nu^{1/2}}\left(1 - (-\eta t_n - \lambda)^{1/2} - \frac{1}{6}(\eta t_n + \lambda) + \cdots\right) \\ &\quad + \frac{1}{\nu}\left[\frac{\frac{1}{2}A^2 - A^2 \log A + \mathscr{D}_1}{2A} \frac{1}{(-\eta t_n - \lambda)^{1/2}}\right. \\ &\quad \left. + \frac{1}{12}A - \frac{1}{6}A \log A + \frac{1}{6}\frac{\mathscr{D}_1}{A} + \cdots\right] \qquad (63) \\ &= \frac{A}{\nu^{1/2}} - \frac{A}{\nu^{1/2}}(-\eta t_n)^{1/2} + \frac{1}{2}\frac{\lambda A}{\nu^{1/2}}\frac{1}{(-\eta t_n)^{1/2}} \\ &\quad - \frac{A\eta t_n}{6(\nu)^{1/2}} + \frac{\frac{1}{2}A^2 - A^2 \log A + \mathscr{D}_1}{2A\nu}\frac{1}{(-\eta t_n)^{1/2}} + \cdots . \end{aligned}$$

The transition expansion is

$$\begin{aligned} u &= \frac{A}{\nu^{1/2}} + \nu^{-5/6}[-A(-\nu^{2/3}\eta t_n)^{1/2} + \cdots] \\ &\quad + \nu^{-7/6}\left[-\frac{1}{6}A\nu^{2/3}\eta t_n + \frac{1}{8}A\frac{\log(-\nu^{2/3}\eta t_n)}{(-\nu^{2/3}\eta t_n)^{1/2}} + \cdots\right] + \cdots \\ &= \frac{A}{\nu^{1/2}} - \frac{A}{\nu^{1/2}}(-\eta t_n)^{1/2} - \frac{1}{6}\frac{A}{\nu^{1/2}}\eta t_n + \frac{1}{8}A\frac{\frac{2}{3}\log \nu}{\nu^{2/3}(-\eta t_n)^{1/2}} + \cdots . \qquad (64) \end{aligned}$$

Comparison of (63) and (64) shows that it is necessary to choose

$$\tfrac{1}{12}A\nu^{-3/2}\log \nu = \tfrac{1}{2}\nu^{-1/2}\lambda A,$$

or $\lambda = \frac{1}{6}(\log\nu/\nu)$, and also

$$\mathscr{D}_1 = A^2 \log A - \tfrac{1}{2}A^2.$$

The matching of the lower transition solution with the jump up occurs as $t^+ \to \omega_0$ and $\tilde{t} \to -\infty$. The intermediate limit is $\nu^{-1} \ll \eta \ll \nu^{-2/3}$, with $t_n < 0$ fixed and $\tilde{t} = \nu\eta t_n$. But since

$$t^+ = \nu^{2/3}\left(\hat{t} - \frac{1}{6}\frac{\log \nu}{\nu}\right) \qquad \text{and} \qquad \tilde{t} = \nu(\hat{t} - \chi(\nu)),$$

we have

$$\tilde{t} = \nu\left(\nu^{-2/3}t^+ + \frac{1}{6}\frac{\log \nu}{\nu} - \chi(\nu)\right), \qquad \text{and} \qquad t^+ - \omega_0 = \nu^{2/3}(\eta t_n + \zeta)$$

where

$$\zeta(\nu) = \chi(\nu) - \frac{1}{6}\frac{\log \nu}{\nu} - \frac{\omega_0}{\nu^{2/3}}.$$

In the intermediate variables the transition expansion is

$$\begin{aligned} u &= \frac{A}{\nu^{1/2}} - \nu^{-5/6}A\left\{\frac{1}{\nu^{2/3}(\eta t_n + \zeta)} - \frac{1}{3}\,\omega_0(\eta t_n + \zeta)\,\nu^{2/3} + \cdots\right\} \\ &\quad + A\nu^{-7/6}\,\frac{\log(-\nu^{2/3}(\eta t_n + \zeta))}{\nu^{2/3}(\eta t_n + \zeta)^2} + \cdots \\ &= \frac{A}{\nu^{1/2}} - \frac{A}{\nu^{2/3}\eta t_n}\left\{1 - \frac{\zeta}{\eta t_n} + \cdots\right\} + \frac{\frac{1}{3}\omega_0 A}{\nu^{1/6}}(\eta t_n + \zeta) + \frac{\frac{2}{3}A\log\nu}{\nu^{5/2}(\eta t_n)^2} + \cdots. \end{aligned} \tag{65}$$

The inner expansion is

$$\begin{aligned} u &= \frac{A}{\nu^{1/2}}\left\{1 + \frac{A}{\nu^{3/2}} + \cdots - \frac{1 + (4A/\nu^{3/2}) + \cdots}{\nu\eta t_n}\right. \\ &\quad \left. + \frac{(1 + (8A/\nu^{3/2}) + \cdots)\log(-\nu\eta t_n)}{(\nu\eta t_n)^2} + \cdots\right\} \\ &\quad + \mu_1(\nu)\,\mathscr{B}_1\left\{\frac{\nu\eta t_n}{3 + \cdots} + \frac{2}{3}\left(1 + \frac{4A}{\nu^{3/2}} + \cdots\right)\log(-\nu\eta t_n) + \cdots\right\} \\ &= \frac{A}{\nu^{1/2}} - \frac{A}{\nu^{3/2}\eta t_n} + \frac{A\log\nu}{\nu^{5/2}(\eta t_n)^2} + \frac{1}{3}\,\mu_1(\nu)\,\mathscr{B}_1\nu\eta t_n + \cdots. \end{aligned} \tag{66}$$

To match (65) and (66) it is necessary to choose $\mu_1\nu = \nu^{1/6}$, that is $\mu_1(\nu) = \nu^{-7/6}$. Further, $\frac{1}{3}\omega_0 A = \frac{1}{3}\mathscr{B}_1$, or $\mathscr{B}_1 = A\omega_0$. Also

$$\frac{A\zeta}{\nu^{3/2}} = \frac{A\log\nu}{\nu^{5/2}}, \qquad \text{or} \qquad \zeta = \frac{\log\nu}{\nu}.$$

Thus,

$$\chi(\nu) = \frac{7}{6}\frac{\log\nu}{\nu} + \frac{\omega_0}{\nu^{2/3}}.$$

This choice of $\mathscr{B}_1$ means that (36) becomes

$$z_1 = A\omega_0\left(1 + \frac{4A}{\nu^{3/2}} + \frac{19A^2}{\nu^3} + \cdots\right) + \text{TST},$$

as $\tilde{t} \to +\infty$.

The only matching remaining to be done is that of the jump down to the lower outer solution and the jump up to the upper outer solution. First, it will be necessary to calculate the expansion of the lower outer solution at the bottom of the jump down; this had been postponed earlier.

To leading order, it was found that as $\hat{t} \to -t_j$,

$$w_0 = \frac{4A^2}{\nu^{3/2}}\left(1 - \frac{8A^2}{\nu^3} - \cdots\right).$$

We expand (13) about this value of w_0 and obtain

$$\begin{aligned} w_0 = \frac{4A^2}{\nu^{3/2}}\left(1 - \frac{8A^2}{\nu^3} + \cdots\right) + \frac{4A^2}{\nu^{3/2}}\left(1 + \frac{24A^2}{\nu^3} + \cdots\right) \\ \times \left[\hat{t} + \tfrac{3}{2}\log\nu - \log A - 2\log 2 - O(\nu^{-3})\right] + \cdots . \end{aligned} \tag{67}$$

Expansion of (14) about the same value of w_0 yields

$$\begin{aligned} w_1 &= \frac{1}{A^2}\left(1 + \frac{16A^2}{\nu^3} + \cdots\right)\Bigg\{\frac{4A^2\mathscr{G}_1}{\nu^{3/2}}\left(1 - \frac{8A^2}{\nu^3} + \cdots\right) \\ &\quad - A^2\left(1 - \frac{8A^2}{\nu^3} + \cdots\right)\frac{4A^2}{\nu^{3/2}} \\ &\quad \times \left(-\frac{3}{2}\log\nu + \log 4A^2 - \frac{8A^2}{\nu^3} - \cdots\right) + \cdots\Bigg\} + \cdots \\ &= \left(1 + \frac{16A^2}{\nu^3} + \cdots\right)\left\{6A^2\frac{\log\nu}{\nu^{3/2}} + \frac{1}{\nu^{3/2}}(4\mathscr{G}_1 - 4A^2\log 4A^2) + O(\nu^{-9/2})\right\} \\ &\quad + \cdots . \end{aligned} \tag{68}$$

The limit to be taken in matching the lower outer solution and the jump down is $\bar{t} \to +\infty$, $t \to 0^+$. We have $\bar{t} = \nu(t - \theta_0)$ and $\hat{t} = t - t_j$, so $\bar{t} = \nu(\hat{t} + t_j - \theta_0)$. We let $\bar{t} = \nu\eta t_\eta$ where $\nu^{-1} \ll \eta \ll 1$ and $t_\eta > 0$ is fixed, so that $\hat{t} = \eta t_\eta - t_j + \theta_0$. In the intermediate variables, the inner expansion is just

$$u = \frac{4A^2}{\nu^2}\left(1 - \frac{8A^2}{\nu^3} + \cdots\right) - \frac{16A^2}{\nu^4}(A^{4/3}\omega_0) + \text{TST}, \tag{69}$$

and the outer expansion is

$$\begin{aligned} u = \frac{4A^2}{\nu^2}\left(1 - \frac{8A^2}{\nu^3} + \cdots\right) + \frac{4A^2}{\nu^2}\left(1 + \frac{24A^2}{\nu^3} + \cdots\right) \\ \times \left[\eta t_\eta - t_j + \theta_0 + \tfrac{3}{2}\log\nu - \log A - 2\log 2 + O(\nu^{-3})\right] + \cdots \\ + \frac{1}{\nu}\left(1 + \frac{16A^2}{\nu^3} + \cdots\right)\Bigg[6A^2\frac{\log\nu}{\nu^{3/2}} + \frac{1}{\nu^{3/2}}(-4A^2\log A \\ - A^2(2 - 8\log 2)) + \cdots\Bigg] + \cdots . \end{aligned} \tag{70}$$

To match these expansions we must pick

$$t_j = \theta_0 + \tfrac{3}{2}\log\nu - \log A - 2\log 2$$
$$+ \frac{3}{2}\frac{\log\nu}{\nu^{1/2}} - \frac{1}{4}(4\log A + (2 - 8\log 2))\frac{1}{\nu^{1/2}} + \cdots. \tag{71}$$

The matching is then correct up to order $\nu^{-5/2}\log\nu$; the first omitted term in Eq. (71) is of order $\nu^{-3}\log\nu$. The error in t_j is smaller than order $\nu^{-1/2}$.

The final matching is of the jump up to the upper outer solution. The jump up ends at $u = \nu^{-1/2}\psi + O(\nu^{-7/6})$; in intermediate variables this is just

$$u = \nu\left(1 - \frac{2A}{\nu^{3/2}} - \frac{2A^2}{\nu^3} + \cdots\right) + O(\nu^{-7/6}) + \mathrm{TST}, \tag{72}$$

and so the outer solution is to be expanded about

$$u_0 = 1 - \frac{2A}{\nu^{3/2}} - \frac{2A^2}{\nu^3} + \cdots.$$

However, this portion of the cycle is no longer given by (8) but by the expression

$$-(u_0 - \tfrac{1}{2})^2 = A^2\bar{\tau},$$

where $\bar{\tau} = \nu^{-2}(\tau - T(\nu))$ and $T(\nu)$ is the period of the oscillation. The outer limit has $\bar{\tau}$ fixed. The variable u_0 takes on the proper value when

$$\bar{\tau} = \bar{\tau}_0 = -\frac{1}{4A^2}\left(1 - \frac{8A}{\nu^{3/2}} + \frac{8A^2}{\nu^3} + \cdots\right).$$

The expansion of u_0 is therefore

$$u_0 = 1 - \frac{2A}{\nu^{3/2}} + \cdots + A_1\left(\bar{\tau} + \frac{1}{4A^2}\left[1 - \frac{8A}{\nu^{3/2}} + \cdots\right]\right)$$
$$+ A_2\left(\bar{\tau} + \frac{1}{4A^2}\left[1 - \frac{8A}{\nu^{3/2}} + \cdots\right]\right)^2 + \cdots,$$

where

$$A_1 = \left.\frac{du_0}{d\bar{\tau}}\right|_{\bar{\tau}=\bar{\tau}_0} = -A^2\left(1 + \frac{4A}{\nu^{3/2}} + \cdots\right),$$

$$A_2 = \left.\frac{d^2u_0}{d\bar{\tau}^2}\right|_{\bar{\tau}=\bar{\tau}_0} = -2A^4\left(1 + \frac{12A}{\nu^{3/2}} + \cdots\right).$$

Then

$$
\begin{aligned}
u_1 &= \frac{1}{u_0 - \frac{1}{2}} \left\{ -u_0 + \frac{1}{2} \log u_0 + \frac{1}{2}(1 + \log 2) \right\} \\
&= \left(\log 2 - 1 + \frac{2A}{\nu^{3/2}} + \cdots \right) \left(1 + \frac{4A}{\nu^{3/2}} + \cdots \right) + \cdots .
\end{aligned}
$$

Now $t = \bar{\tau}\nu^2 + T(\nu)$ and

$$
\begin{aligned}
\tilde{t} &= \nu(\acute{t} - \chi(\nu)) \\
&= \nu \left(t - t_J - \frac{7}{6} \frac{\log \nu}{\nu} - \frac{\omega_0}{\nu^{2/3}} \right) \\
&= \nu \left(t - \frac{\omega_0}{A^{2/3}} - \frac{3}{2} \log \nu + \log A + 2 \log 2 + \cdots \right) .
\end{aligned}
$$

Thus

$$
\tilde{t} = \nu \left(\bar{\tau}\nu^2 + T(\nu) - \frac{3}{2} \log \nu + \log A - \frac{\omega_0}{A^{2/3}} + 2 \log 2 + \cdots \right) .
$$

We choose the intermediate limit $\nu^{-1} \ll \eta \ll \nu^2$, where $\tilde{t} = \nu\eta t_\eta$ and t_η is fixed. Thus,

$$
\bar{\tau}\nu^2 = \eta t_\eta - T(\nu) + \frac{3}{2} \log \nu - \log A + \frac{\omega_0}{A^{2/3}} - 2 \log 2 + \cdots .
$$

In the intermediate variables the outer expansion is

$$
\begin{aligned}
u = \nu &\left(1 - \frac{2A}{\nu^{3/2}} + \cdots \right) - \nu A^2 \left(1 + \frac{4A}{\nu^{3/2}} + \cdots \right) \\
&\times \left\{ \frac{\eta t_\eta}{\nu^2} - \frac{T(\nu) - \frac{1}{3} \log \nu + \log A - A^{-2/3}\omega_0 + 2 \log 2}{\nu^2} \right. \\
&\left. + \frac{1}{4A^2} \left(1 - \frac{8A}{\nu^{3/2}} + \cdots \right) \right\} + \cdots + \left(\log 2 - 1 + \frac{2A}{\nu^{3/2}} + \cdots \right) \\
&\times \left(1 + \frac{4A}{\nu^{3/2}} + \cdots \right) .
\end{aligned} \tag{73}
$$

In order to match (72) and (73) we choose

$$
\begin{aligned}
T(\nu) = \frac{\nu^2}{4A^2} &\left(1 - \frac{8A}{\nu^{3/2}} + \cdots \right) + \frac{3}{2} \log \nu - \log A \\
&+ \frac{\omega_0}{A^{2/3}} - 2 \log 2 + \frac{\nu}{A^2}(1 - \log 2).
\end{aligned}
$$

The error made in $T(\nu)$ by omitting other terms in u_0 and u_1 is $O(\nu^{-1})$, while the error made by omitting u_2 is $O(1)$.

10. Summary

The more detailed description of the solution of (3) is now given by the Eqs. (8), (9), (39), (46), (22), (27), (13), (14), (51), (55), (31), and (34), together with the values of the integration constants that were found from the matching. It is clear how the calculation could be continued if it were desired.

For the values $\nu = 78$, $A = 8.2$, $\omega_0 = 2.338$ we calculate $T(\nu) = 24.5$. Lavenda, Nicolis, and Herschkowitz–Kaufman [2] calculate $T = 24.3$ asymptotically and $T = 25.3$ numerically. It seems clear that it is not ν alone, but some combination of ν and A that is the proper large parameter in this problem. However, A enters into the formulae for $T(\nu)$ and u in a very complicated way, and it is not clear what the combination should be. The fact that the error made in $T(\nu)$ by omitting u_2 is $O(1)$ indicates that it is partly good luck that the asymptotic values of T agree as well as they do with the real value. The other number that the authors calculate in [2] is the maximum of $X = A/u$. This corresponds to the bottom of the jump down, which is given by (62) as

$$u_{\min} = \frac{4A^2}{\nu^2}\left(1 - \frac{4A^{4/3}\omega_0}{\nu^2} - \frac{8A^2}{\nu^3} + \cdots\right).$$

Thus,

$$X_{\max} = \frac{\nu^2}{4A}\left(1 + \frac{4A^{4/3}\omega_0}{\nu^2} + \frac{8A^2}{\nu^3} + \cdots\right) = 190.5.$$

Lavenda, Nicolis, and Herschkowitz–Kaufman calculate $X_{\max} = 189$ numerically.

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References

1. J. D. Cole, "Perturbation Methods in Applied Mathematics," Blaisdell, Waltham, Mass., 1968.
2. B. Lavenda, G. Nicolis, and M. Herschkowitz-Kaufman, Chemical instabilities and relaxation oscillations, *J. Theor. Biol.* **32** (1971), 283–292.
3. R. Lefever and G. Nicolis, Chemical instabilities and sustained oscillations, *J. Theor. Biol.* **30** (1971), 267–284.
4. N. Levinson and O. K. Smith, A general equation for relaxation oscillations, *Duke Math. J.* **9** (1942), 382–403.

5. YE. F. MISHCHENKO, Asymptotic calculation of periodic solutions of systems of differential equations containing small parameters in the derivatives, *Amer. Math. Soc. Transl. Ser.* 2 **18** (1961), 199–230.
6. L. S. PONTRYAGIN, Asymptotic behaviour of the solutions of systems of differential equations with a small parameter in the higher derivatives, *Amer. Math. Soc. Transl. Ser.* 2 **18** (1961), 295–320.
7. J. A. BOA AND D. S. COHEN, Bifurcation of localized disturbances in a model biochemical reaction, *SIAM J. Appl. Math.* **30** (1976), 123–135.